Wave-equation migration velocity analysis with time-lag imaging

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ABSTRACT

Wave-equation migration velocity analysis (WEMVA) is a technique designed to extract velocity information from migrated images. The velocity updating is done by optimizing the coherence of images migrated with the known background velocity model. Its capacity for handling multipathing makes it appropriate in complex subsurface regions characterized by strong velocity variation. WEMVA operates by establishing a linear relation between a slowness perturbation and a corresponding image perturbation. The linear relationship is derived from conventional extrapolation operators and it inherits the main properties of frequency-domain wavefield extrapolation. A key step in implementing WEMVA is to design an appropriate procedure for constructing an image perturbation relative to an image reference that represents difference between current image and a true, or more correct image of the subsurface geology. The target of the inversion is to minimize such an image perturbation. Using time-shift common-image gathers, one can characterize the imperfections of migrated images by defining the focusing error as the shift of the focus of reflections along the time-shift axis. Under the linear approximation, the focusing error can be transformed into an image perturbation by multiplying it with an image derivative taken relative to the time-lag parameter. As the focusing error is caused by the incorrect velocity model, the resulting image
perturbation can be considered as a mapping of the velocity model error in the image space. The inversion is then performed to produce velocity updates. Such an approach for constructing the image perturbation is computationally efficient and simple to implement. Synthetic examples demonstrate the successful application of our method to a layers model and a subsalt velocity update problem.
INTRODUCTION

In regions characterized by complex subsurface structure, prestack wave-equation depth migration, i.e. one-way wave-equation migration (WEM) or reverse-time migration (RTM), is a powerful tool for accurately imaging the earth’s interior (Gray et al., 2001; Etgen et al., 2009). However, the quality of the final image greatly depends on the accuracy of the velocity model. Thus, a key challenge for seismic imaging in complex geology is an accurate determination of the velocity model in the area under investigation (Symes, 2008; Woodward et al., 2008; Virieux and Operto, 2009).

Migration velocity analysis (MVA) generally refers to tomographic methods implemented in the image domain. These techniques are based on the principle that the quality of the seismic image is optimized when the data are migrated with the correct velocity model. MVA updates the velocity model by optimizing the properties measuring the coherence of migrated images. As a result, the quality of both the image and velocity model is improved by MVA.

Typically, the input for MVA is represented by various types of common-image gathers (CIGs), e.g. shot-domain CIG (Al-Yahya, 1989), surface-offset CIG (Mulder and ten Kroode, 2002), subsurface-offset CIG (Rickett and Sava, 2002), time-shift CIG (Sava and Fomel, 2006), angle-domain CIG (Sava and Fomel, 2003; Biondi and Symes, 2004), or space- and time-lag extended-image gathers (Yang and Sava, 2010; Sava and Vasconcelos, 2010). Different kinds of CIGs determine whether semblance or focusing analysis should be used to measure the coherence of the image.

In practice, there are many possible approaches for implementing MVA with different CIGs. However, all such realizations share the common element that they need a carrier of information
to connect the input CIGs to the output velocity model. Thus, one can categorize MVA techniques into ray-based and wavefield-based methods in terms of the information carrier they use. Ray-based methods, which are often described as traveltime MVA, refer to techniques using wide-band rays as the information carrier (Bishop et al., 1985; Al-Yahya, 1989; Stork and Clayton, 1991; Woodward, 1992; Stork, 1992; Liu and Bleistein, 1995; Jiao et al., 2002). Wavefield-based MVA methods refer to techniques using band-limited wavefields as the information carrier (Biondi and Sava, 1999; Mulder and ten Kroode, 2002; Shen and Calandra, 2005; Soubaras and Gratacos, 2007; Xie and Yang, 2008). Generally speaking, ray-based methods have the advantage over wavefield-based methods in that they involve simple implementation and efficient computation. In contrast, wavefield-based methods are capable of handling complicated wave propagation phenomena which always occur in complex subsurface regions. Therefore, they are more robust and consistent with the wavefield-based migration techniques used in such regions. Furthermore, the resolution of wavefield-based methods is higher than that of ray-based methods because they employ fewer approximations to wave propagation.

Different from traveltime MVA, wavefield-based MVA requires an image residual as the input, which is equivalent to the data misfit defined in the image domain. Wave-equation MVA (WEMVA) (Sava and Biondi, 2004a,b) and differential semblance optimization (WEDSO) (Shen et al., 2003; Shen and Symes, 2008) are two common approaches for wavefield-based MVA. The image residual for WEMVA and WEDSO are obtained either by constructing a linearized image perturbation (Sava et al., 2005) or by applying a penalty operator to offset- or angle-domain CIGs (Symes and Carazzone, 1991; Shen and Calandra, 2005), respectively.

Focusing analysis is a commonly used method for refining the velocity model (Faye and Jeannot, 1986). It evaluates the coherence of migrated images by measuring the focusing of reflections. The focusing information can be extracted from time-shift CIG (Sava and Fomel, 2006), and it is
quantified as the shift of the focus of reflections along the time-shift axis from the origin. Such a shift is often defined as focusing error and indicates the existence of the velocity model error (MacKay and Abma, 1992; Nemeth, 1995). Therefore, the focusing error can be used for velocity model optimization. Wang et al. (2005) propose a tomographic approach using focusing analysis for re-datuming data sets. The approach is a ray-based method and thus may become unstable when the multi-pathing problem exists due to the complex geology. Higginbotham and Brown (2008) and Brown et al. (2008) also propose a method to convert this focusing error into velocity updates using an analytic formula. This approach is based on 1D assumptions and the measured focusing error is transformed into vertical updates only. Hence, the method become less accurate in models with complex subsurface environments. Nevertheless, (Wang et al., 2008) and Wang et al. (2009) illustrate that focusing analysis can effectively improve the image quality and be used for updating the velocity model in subsalt areas.

For the implementation of WEMVA, one important component is the construction of an image perturbation which is linked directly to a velocity perturbation. To construct the image perturbation, one first needs to measure the coherence of the image. Sava and Biondi (2004b) discuss two types of measurement for the imperfections of migrated images, i.e. focusing analysis (MacKay and Abma, 1992; Lafond and Levander, 1993) and moveout analysis (Yilmaz and Chambers, 1984; Biondi and Sava, 1999). The most common approach for constructing the image perturbation is to compare a reference image with its improved version. However, such an image-comparison approach has at least two drawbacks. First, the improved version of the image is always obtained by re-migration with one or more models, which is often computationally expensive. Second, if the reference image is incorrectly constructed, the difference between two images might exceed the small perturbation assumption. This can lead to cycle skipping which changes the convergence properties of the inversion. An alternative to this approach, discussed in Sava and Biondi (2004a),
uses prestack Stolt residual migration (Stolt, 1996; Sava, 2003) to construct a linearized image perturbation. This alternative approach avoids the cycle skipping problem, but suffers from the approximation embedded in the underlying Stolt migration.

In this paper, we propose a new methodology of constructing a linearized image perturbation for WEMVA based on the time-shift imaging condition and focusing analysis. We demonstrate that such an image perturbation can be easily computed by a simple multiplication of the image derivative constructed based on a reference image and the measured focusing error. We also demonstrate that this type of image perturbation is fully consistent with the linearization embedded in WEMVA. Finally, we illustrate the feasibility of our approach by testing the method on simple and complex synthetic data sets.

**THEORY**

Under the single scattering approximation, seismic migration consists of two steps: wavefield reconstruction followed by the application of an imaging condition. We commonly discuss about a “source” wavefield, originating at the seismic source and propagating in the medium prior to any interaction with the reflectors, and a “receiver” wavefield, originating at discontinuities and propagating in the medium to the receivers (Berkhout, 1982; Claerbout, 1985). The two wavefields are kinematically equivalent at discontinuities of material properties. Any mismatch between the wavefields indicates inaccurate wavefield reconstruction typically assumed to be due to inaccurate velocity. The source and receiver wavefields $u_s$ and $u_r$ are four-dimensional objects as functions of position $x = (x, y, z)$ and frequency $\omega$,

\begin{align}
    u_s &= u_s(x, \omega), \\
    u_r &= u_r(x, \omega).
\end{align}
An imaging condition is designed to extract from these extrapolated wavefields the locations where reflections occur in subsurface. A conventional imaging condition (Claerbout, 1985) forms an image as the zero cross-correlation lag between the source and receiver wavefields:

\[ r(x) = \sum_{\omega} \overline{u_s(x, \omega)} u_r(x, \omega), \]  

(3)

where \( r \) is the image of subsurface and overline represents complex conjugation. An extended imaging condition (Sava and Fomel, 2006) extracts the image by cross-correlation between the wavefields shifted by the time-lag \( \tau \):

\[ r(x, \tau) = \sum_{\omega} u_s(x, \omega) \overline{u_r(x, \omega)} e^{2i\omega\tau}. \]  

(4)

Other possible extended imaging conditions include space-lag extension (Rickett and Sava, 2002) or space- and time-lag extensions (Sava and Vasconcelos, 2010), but we do not discuss these types of imaging conditions here.

In the process of seismic migration, if the source and receiver wavefields are extrapolated with the correct velocity model, the result of cross-correlation between reconstructed wavefields, i.e. the migrated image, is maximized at zero time-lags. In other words, if the velocity used for migration is correct, reflections in an image are focused at zero offset and zero time. If the source and receiver wavefields are extrapolated with the incorrect velocity model, the result of cross-correlation is not maximized at zero lags. As a consequence, reflections in an image are focused at nonzero time but zero offset. This indicates the existence of error for downward continuation of the reconstructed wavefields. In such a situation, we can apply focusing analysis and extract the information about the velocity model. A commonly used approach for focusing analysis is to measure the focusing error in either depth domain or time domain. The depth-domain focusing error is defined as the
depth difference between focusing depth $d_f$ and migration depth $d_m$ (MacKay and Abma, 1992)

$$\Delta z = d_f - d_m .$$

(5)

$$d_f = \frac{d}{\rho} ,$$

(6)

and

$$d_m = d\rho .$$

(7)

Here $d$ is the true depth of the reflection point, $\rho = \frac{V_m}{V}$ is the ratio between migration and true velocities. The time-domain focusing error is defined as the time-shift being applied to the reconstructed wavefields to achieve focusing of reflections. Yang and Sava (2010) quantitatively analyze the influence of the velocity model error on focusing property of reflections, and derive the formula connecting depth-domain and time-domain focusing error:

$$\Delta \tau = \frac{d_f - d_m}{V_m} ,$$

(8)

where $V_m$ represents the migration velocity. Notice that the formulae for $d_f$ and $d_m$ are derived under the assumptions of constant velocity, small offset angle and horizontal reflector. As a consequence, the analytic formula in equation 8 is an approximation of the true focusing error, and has limited applicability in practice.

To use WEMVA for velocity optimization, one needs to start from linearization of the problem since an image is usually a nonlinear function of the velocity model. Furthermore, a nonlinear optimization problem is more difficult to solve than a linear optimization problem. We represent the
true slowness as the sum of a background slowness $s_b$ and a slowness perturbation $\Delta s$:

$$s(x) = s_b(x) + \Delta s(x).$$ \hfill (9)

Likewise, the image can also be characterized as a sum of a background image $r_b$ and an image perturbation $\Delta r$:

$$r(x) = r_b(x) + \Delta r(x).$$ \hfill (10)

The perturbation $\Delta s$ and $\Delta r$ are related by the wave-equation tomographic operator $L$ derived from the linearization of one-way wave-equation migration operator, as demonstrated by Sava and Biondi (2004a). A brief summary of derivation and construction of $L$ can be found in appendix A. Consequently, we can establish a linear relationship between the image perturbation and slowness perturbation:

$$\Delta r = L\Delta s.$$ \hfill (11)

Sava and Vlad (2008) further illustrates the implementation of operator $L$ for different imaging configurations, i.e. zero-offset, survey-sinking, and shot-record cases.

The focusing information of $\Delta \tau$ is available on the extended images constructed with equation 4. In other words, the focus of reflections can be evaluated by locating the maximum energy of reflection events in time-shift CIGs. If the focus is located at zero time shift, the velocity model is correct. Otherwise, the distance of the focus from origin is measured along the time-shift axis and defined as the focusing error. Then, an improved image $\tilde{r}$ can be obtained by choosing the image from the time-shift CIG at the $\tau$ value corresponding to the focusing error:

$$\tilde{r}(x) = r(x, \tau)|_{\tau = \Delta \tau},$$ \hfill (12)
and an image perturbation can be constructed by a direct subtraction:

$$\Delta r(x) = \tilde{r}(x) - r_b(x).$$  \hspace{1cm} (13)

Although such an approach is straightforward, the constructed image perturbation might be phase-shifted too much with respect to the background image $r_b$ and thus violates the Born approximation required by the tomographic operator $L$. To overcome the challenge, Sava and Biondi (2004a) propose to construct a linearized image perturbation as

$$\Delta r(x) \approx K'_{\rho=1}[r_b] \Delta \rho,$$  \hspace{1cm} (14)

where $K$ is the prestack Stolt residual migration operator (Stolt, 1996; Sava, 2003), the $'$ sign represents derivation relative to the velocity ratio $\rho$, and $\Delta \rho = \rho - 1$. Similarly, we can also construct an image perturbation by a linearization of the image relative to the time-shift parameter $\tau$:

$$\Delta r(x) \approx \left. \frac{\partial r(x,\tau)}{\partial \tau} \right|_{\tau=0} \Delta \tau,$$  \hspace{1cm} (15)

where the image derivative with respect to time-shift $\tau$ is

$$\frac{\partial r(x,\tau)}{\partial \tau} = \sum_\omega (2i\omega) \frac{u_s(x,\omega)}{u_r(x,\omega)} u_r(x,\omega) e^{2i\omega \tau}.$$  \hspace{1cm} (16)

Comparing two different approaches for constructing the image perturbation in equation 14 and equation 16, they share a similar form of the formulae. The differences lie in how to construct the image derivative and how to measure the quantity relating to the velocity model error. The linearization of both the approaches is consistent with the characteristics of the tomographic operator $L$. 

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One can follow the procedure outlined below to construct the image perturbations for WEMVA:

- Migrate the image and output time-shift CIGs according to equation 4;
- Measure $\Delta \tau$ on time-shift CIG panels by picking the focus, i.e. maximum energy, for all reflections;
- Construct the image derivative according to equation 16;
- Construct the linearized image perturbation according to equation 15.

After we construct the image perturbation $\Delta r$, we can solve for the slowness perturbation $\Delta s$ by minimizing the objective function,

$$ J(\Delta s) = \frac{1}{2} \| \Delta r - L \Delta s \|^2. \quad (17) $$

Such an optimization problem can be solved iteratively using conjugate-gradient-based methods. As most practical inverse problems are ill-posed, additional constraints must be imposed during the inversion to obtain a stable and convergent result. This can be done by adding a model regularization term in the objective function equation 17, leading to the modified objective function:

$$ J(\Delta s) = \frac{1}{2} \| \Delta r - L \Delta s \|^2 + \alpha^2 \| A \Delta s \|^2, \quad (18) $$

where $\alpha$ is a scalar to control the relative weights between data residual and model norm and $A$ is a regularization operator (Fomel, 2007). How to chose $A$ and $\alpha$ is outside the scope of discussion in this paper.
EXAMPLES

We illustrate our methodology with two synthetic examples. The first example consists of four
dipping layers with different thickness and dipping angle. The velocities of the four layers are 1.5,
1.6, 1.7, and 1.8 km/s, respectively. The background model is constant with velocity of 1.5 km/s.

Figure 1(a) and 1(b) show the true and background velocity models, respectively. Figure 2(a) and
2(b) are the image migrated with the true and background velocity model, respectively. Comparing
these two images, one can observe that the second and third reflectors in Figure 2(b) are positioned
incorrectly due to the error of the velocity model.

Figure 3(a) - 3(c) shows three time-shift CIGs migrated with the background model. The gathers
are chosen at $x = 1.2$, $x = 1.8$, and $x = 2.3$ km, respectively. The picking is done by applying
an automatic picker to the envelope of the original time-shift CIGs. The solid line in the figure
plots the picked focusing error, while the dashed line represents zero time shift. In this case, as
the velocity errors for the second and third layers increase with depth, the focusing error increases
with depth as well. Therefore, bigger shifts can be observed for deeper events. Figure 4(a) shows
the focusing error as a function of position and depth. This figure depicts the focusing error picked
from the time-shift CIGs constructed at every horizontal location $x$. Since the layers have different
thickness, the associated velocity errors change as a function of horizontal and vertical directions.
Therefore, a spatially varying focusing error of the subsurface is observed.

We construct the image perturbation using the procedure discussed in the preceding sections.
Next, we perform the inversion based on the objective function in equation 18. The inverse problem
is solved using the conjugate-gradient method. For this simple model, two nonlinear iterations are
enough to reconstruct the velocity model.

Figure 1(c) shows the updated model after the inversion. The velocity for the second and third
layer are correctly inverted. Here, the velocity update occurs in the region above the fourth layer. This is due to the fact that no reflection exists in the bottom part of the model, and the input image perturbation carries no velocity information for the bottom layer. Thus, the inversion does not compute velocity updates in the area. Figure 2(c) shows the image migrated with the updated model. The reflections are positioned in the correct depth, as compared with Figure 2(a).

Figure 3(d) - 3(f) shows time-shift CIGs obtained by migration with the updated model. The gathers are chosen at the same positions as the gathers shown in Figure 3(a) - 3(c), i.e. at $x = 1.2$, $x = 1.8$, and $x = 2.3$ km, respectively. The focus of all the reflections are located at zero time-shift $\tau$, as demonstrated by the fact that the solid and dashed are overlapped with each other in the gather. Figure 4(b) is the focusing error panel of the subsurface. The value of the focusing error is zero almost everywhere. This means that the inversion has achieved the target of minimizing the focusing error and confirms the successful reconstruction of the velocity model in this example.

We also apply our methodology to a subsalt velocity update example. The target area is chosen from the subsalt portion of the Sigsbee 2A model (Paffenholz et al., 2002), ranging from $x = 9.5$ km to $x = 18.5$ km, and $z = 5.0$ km to $z = 9.3$ km. For this example, the main goals are to image correctly the fault located between $x = 14.0$ km to $x = 16.0$ km, and $z = 6.0$ km to $z = 9.0$ km, to increase the focusing for the deeper diffractors, and to positioning correctly the bottom flat reflector. We assume correct knowledge of the velocity above the salt and known salt geometry. The background model for the target zone is obtained by scaling the true model with a constant factor 0.1. The true and background velocity model are depicted in Figure 5(a) and 5(b), respectively. Figure 6(a) and 6(b) show the images migrated with the true and background model, respectively. Due to the error of the background velocity model, the fault is obscured by nearby sediment reflections, the deeper diffractions are defocused, and the bottom flat reflector is positioned far away from its correct depth.
Figure 7(a) - 7(c) show several time-shift CIGs migrated with the background model at $x = 13.0$, $x = 14.8$, and $x = 16.6$ km, respectively. The picked focusing error is directly overlain on the gathers, although the picking is done on the envelope of the original time-shift CIGs just as the previous example. Here, we notice that the picks do not start at zero on the top of the gathers, although the associated focusing error should be zero. Such an error is mainly caused by the truncation of reflections in the gathers. Figure 8(a) shows the focusing error in the target area. Notice that variation exists for the picked focusing error in lateral and vertical directions. Part of such a variation is caused by the uneven illumination of the subsalt area, which leads to difficulties for the automatic picker.

Next, we use the same procedure to construct the image perturbation as introduced in the preceding section. Then we run the inversion to obtain the velocity update by minimizing the objective function in equation 18. Figure 5(c) shows the updated model after three nonlinear iterations. The inversion has correctly reconstructed the velocity model in the subsalt area. Figure 6(c) shows the image migrated with the updated model. The fault located between $x = 14.0$ km to $x = 16.0$ km, and $z = 6.0$ km to $z = 9.0$ km is delineated and clearly visible in the image. The bottom reflector is positioned at the correct depth, and become more coherent. The deeper diffractions are also well focused. These improvements in the image imply a correct update of the velocity model.

Figure 7(d) - 7(f) plot several time-shift CIGs migrated with the background model at $x = 13.0$, $x = 14.8$, and $x = 16.6$ km, respectively. The focusing error picks are overlain on the gathers. Most reflections are focused at zero time shift, which means the focusing error is reduced thanks to the more accurate velocity model after the inversion. Figure 8(b) plot the focusing error in the target area after velocity update and remigration. The focusing error is reduced after the inversion, which indicates the successful optimization of the velocity model.
To further confirm that the velocity model is correctly inverted, we also compute angle-domain CIGs with the background and updated models and use the flatness of reflection events in the gathers as the criterion to evaluate the result of inversion. The angle-domain CIGs are chosen at the same locations at the time-shift CIGs. Figure 9(a) - 9(c) plot the gathers corresponding to the background velocity model, while Figure 9(d) - 9(f) show the gathers for the updated velocity model. In comparison, the reflection events in the gathers obtained with the updated model are more flat than the events in the gathers obtained with the background model. This demonstrates the improvement of the quality of the velocity model after the optimization. However, we notice that the curvature of the residual moveout in the gathers obtained with the background model is not easily distinguishable. This can cause difficulties for curvature picking and thus degrades the velocity estimation methods relying on flattening the residual moveout. In such a situation, minimizing the focusing error, as in the case of our approach, may provide a reliable alternative for velocity optimization.

**DISCUSSION**

The construction of the linearized image perturbation using time-shift imaging condition and focusing analysis is a good replacement for the approach using Stolt residual migration. Compared with the procedure based on Stolt residual migration, our method has the main advantage of a low computational cost, since the computation of the linearized image perturbation adds just a trivial cost to the construction of time-shift CIGs. More importantly, no expensive re-migration scan and angle decomposition are required in this procedure. Furthermore, the method based on time-shift CIGs is computationally more attractive in 3D application as only one additional dimension is required for constructing CIGs, while two additional dimensions are needed for constructing lag-domain CIGs (Shen and Symes, 2008).
The inversion scheme described in the paper is a wave-equation-based tomographic approach. Although the information is extracted by applying focusing analysis to time-shift CIGs, the process of velocity updating does not rely on any analytic formulae as in the case for conventional depth-focusing analysis. Therefore, the technique discussed in this paper has applicability to models with arbitrary lateral velocity variations.

Finally, we emphasize that picking the focusing error is extremely important for our approach as the focusing error determines the direction and magnitude of the velocity update. Therefore, the process of the picking should be carefully implemented.

**CONCLUSIONS**

We develop a new method for implementing WEMVA based on time-lag imaging and focusing analysis. The objective of the velocity optimization is to minimize the focusing error measured from time-shift CIGs. The methodology relies on constructing linearized image perturbations by applying focusing analysis to time-shift CIGs. The focusing error is defined as the shift of the focus for reflections along the time-shift axis and provides a measurement for the accuracy of the velocity model used in migration. We use this information in conjunction with image derivatives relative to the time-shift parameter to compute linearized image perturbations. The image perturbation obtained by this approach is consistent with the Born approximation used for the WEMVA operators. Compared with the conventional approach for constructing linearized image perturbation using Stolt residual migration, our approach is efficient, since the main cost is just the construction of the time-shift CIG and is much lower than the cost of re-migration scans and angle decomposition. In addition, our method is accurate, since it does not make use of Stolt-like procedures which incorporate strong assumptions about the smoothness of the background model. Thus, our approach...
is suitable for the areas with complex subsurface structure.

The synthetic examples demonstrate the validity of the linearized image perturbation constructed using our new method. The inversion with such an image perturbation as the input can render satisfactory updates for the velocity model. The success of the velocity optimization is confirmed by the fact that the focusing error is reduced after the inversion.

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APPENDIX A

Wave-equation tomographic operator

The construction of wave-equation tomographic operator $L$ used in equation 11 starts from separating the total slowness $s$ into a known background $s_b$ and a slowness perturbation $\Delta s$, as shown in equation 9. Next, we linearize the depth wavenumber $k_z$ relative to the background slowness $s_b$ using a truncated Taylor expansion series expansion

$$k_z \approx k_{zb} + \frac{\partial k_z}{\partial s} \bigg|_{s_b} \Delta s (x), \quad (A-1)$$

where the depth wavenumber in the background medium characterized by slowness $s_b (x)$ is

$$k_{zb} = \sqrt{(2\omega s_b (x))^2 - |k_x|^2}. \quad (A-2)$$

A wavefield perturbation $\Delta u (x)$ caused at depth $z + \Delta z$ by a slowness perturbation $\Delta s (x)$ at depth $z$ is obtained by subtraction of the wavefields extrapolated from $z$ to $z + \Delta z$ in the true and background models:

$$\Delta u_{z+\Delta z} (x) = e^{-ik_z\Delta z} u_z (x) - e^{-ik_{z0}\Delta z} u_z (x)$$

$$= e^{-ik_{z0}\Delta z} \left[ e^{-i \frac{\partial k_z}{\partial s} \bigg|_{s_b} \Delta s (x) \Delta z} - 1 \right] u_z (x). \quad (A-3)$$

Here, $\Delta u (x)$ and $u (x)$ correspond to a given depth level $z$ and frequency $\omega$. A similar relation can be applied at all depths and all frequencies.

Equation A-3 establishes a non-linear relation between the wavefield perturbation $\Delta u (x)$ and the slowness perturbation $\Delta s (x)$. Given the complexity and cost of numeric optimization based on
non-linear relations between model and wavefield parameters, it is desirable to simplify this relation
to a linear relation between model and data parameters. Assuming small slowness perturbations, i.e.
small phase perturbations, the exponential function $e^{\pm i \frac{\partial k_z}{\partial s} |_{sb} \Delta s(x) \Delta z}$ can be linearized using the
approximation $e^{i\phi} \approx 1 + i\phi$ which is valid for small values of the phase $\phi$. Therefore, the wavefield
perturbation $\Delta u(x)$ at depth $z$ can be written as

$$\Delta u(x) \approx -i \frac{\partial k_z}{\partial s} |_{sb} \Delta z \, u(x) \, \Delta s(x)$$
$$\approx -i \Delta z \frac{2\omega u(x) \Delta s(x)}{\sqrt{1 - \left( \frac{|k_z|}{\omega s_b(x)} \right)^2}}.$$  (A-4)

In the case of shot-record migration, we have both the source and receiver wavefields. Thus, the
wavefield perturbations for both the source and receiver can be computed by a direct generalization
of equation A-4.

$$\Delta u_s(x) \approx +i \frac{\partial k_z}{\partial s} |_{sb} \Delta z \, u_s(x) \, \Delta s(x)$$
$$\approx +i \Delta z \frac{\omega u_s(x) \Delta s(x)}{\sqrt{1 - \left( \frac{|k_z|}{\omega s_b(x)} \right)^2}},$$  (A-5)

and

$$\Delta u_r(x) \approx -i \frac{\partial k_z}{\partial s} |_{sb} \Delta z \, u_r(x) \, \Delta s(x)$$
$$\approx -i \Delta z \frac{\omega u_r(x) \Delta s(x)}{\sqrt{1 - \left( \frac{|k_z|}{\omega s_b(x)} \right)^2}},$$  (A-6)

The image perturbation at depth $z$ is obtained from the source and receiver scattered wavefields.
using the relation

\[
\Delta r (x) = \sum_{\omega} \left( \overline{u_s (x, \omega)} \Delta u_r (x, \omega) + \overline{\Delta u_s (x, \omega)} u_r (x, \omega) \right),
\]

(A-7)

which corresponds to the frequency-domain zero-lag cross-correlation of the source and receiver wavefields.

The tomographic operator \( L \) involves computing the wavefield perturbations using equation A-5 and equation A-6 and the image perturbation using equation A-7. Therefore, given a slowness perturbation \( \Delta s \), the relating image perturbation is constructed via the operator \( L \). A more detailed implementation of the operator \( L \) and its adjoint can be found in Sava and Vlad (2008).
LIST OF FIGURES

1 Velocity profiles of the layers model. (a) The true velocity model. (b) The background velocity model with a constant velocity of the first layer 1.5 km/s. (c) The updated velocity model.

2 Images migrated with (a) the true velocity model, (b) the background velocity model, and (c) the updated velocity model.

3 Time-shift CIGs migrated with the background model (a) at $x = 1.2$ km, (b) at $x = 1.8$ km, and (c) at $x = 2.3$ km. Time-shift CIGs migrated with the updated model at (d) $x = 1.2$ km, (e) at $x = 1.8$ km, and (f) at $x = 2.3$ km. The overlain solid lines are picked focusing error. The dash line represents zero time shift.

4 (a) Focusing error panel corresponding to the background model. (b) Focusing error panel corresponding to the updated model.

5 Velocity profiles of Sigsbee 2A model. (a) The true velocity model, (b) the background velocity model, and (c) the updated model.

6 Images migrated with (a) the true velocity model, (b) the background velocity model, and (c) the updated model.

7 Time-shift CIGs migrated with the background model, (a) at $x = 13.0$ km, (b) at $x = 14.8$ km, and (c) at $x = 16.6$ km. Time-shift CIGs migrated with the updated model, (d) at $x = 13.0$ km, (e) at $x = 14.8$ km, and (f) at $x = 16.6$ km. The overlain solid lines are picked focusing error. The dash line represents zero time shift.

8 (a) Focusing error panel corresponding to the background model. (b) Focusing error panel corresponding to the updated model.

9 Angle-domain CIGs migrated with the background model, (a) at $x = 13.0$ km, (b) at $x = 14.8$ km, and (c) at $x = 16.6$ km. Angle-domain CIGs migrated with the updated model, (d) at $x = 13.0$ km, (e) at $x = 14.8$ km, and (f) at $x = 16.6$ km.
Figure 1: Velocity profiles of the layers model. (a) The true velocity model. (b) The background velocity model with a constant velocity of the first layer 1.5 km/s. (c) The updated velocity model.
Figure 2: Images migrated with (a) the true velocity model, (b) the background velocity model, and (c) the updated velocity model.
Figure 3: Time-shift CIGs migrated with the background model (a) at $x = 1.2$ km, (b) at $x = 1.8$ km, and (c) at $x = 2.3$ km. Time-shift CIGs migrated with the updated model at (d) $x = 1.2$ km, (e) at $x = 1.8$ km, and (f) at $x = 2.3$ km. The overlain solid lines are picked focusing error. The dash line represents zero time shift.
Figure 4: (a) Focusing error panel corresponding to the background model. (b) Focusing error panel corresponding to the updated model.
Figure 5: Velocity profiles of Sigsbee 2A model. (a) The true velocity model, (b) the background velocity model, and (c) the updated model.
Figure 6: Images migrated with (a) the true velocity model, (b) the background velocity model, and (c) the updated model.
Figure 7: Time-shift CIGs migrated with the background model, (a) at $x = 13.0$ km, (b) at $x = 14.8$ km, and (c) at $x = 16.6$ km. Time-shift CIGs migrated with the updated model, (d) at $x = 13.0$ km, (e) at $x = 14.8$ km, and (f) at $x = 16.6$ km. The overlain solid lines are picked focusing error. The dash line represents zero time shift.
Figure 8: (a) Focusing error panel corresponding to the background model. (b) Focusing error panel corresponding to the updated model.
Figure 9: Angle-domain CIGs migrated with the background model, (a) at $x = 13.0$ km, (b) at $x = 14.8$ km, and (c) at $x = 16.6$ km. Angle-domain CIGs migrated with the updated model, (d) at $x = 13.0$ km, (e) at $x = 14.8$ km, and (f) at $x = 16.6$ km.