Adaptive phase-ray wavefield extrapolation

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SUMMARY

Riemannian wavefield extrapolation (RWE) is a generalization of downward continuation to coordinate systems that closely conform to the orientation of extrapolated wavefields. If the coordinate system overturns, so does the computed wavefield, despite being extrapolated with a one-way solution to the acoustic wave-equation. This allows for accurate imaging of structures with arbitrarily large dips using simple operators which are equivalent to the standard 15° extrapolators. An obvious question for RWE is which coordinate system is optimal for a given velocity model. The solution advocated in this paper is a recursive bootstrap procedure where a frequency-dependent coordinate system is computed on the fly at every step from the gradient of the monochromatic wavefield phase of the preceding few steps, coupled with standard RWE.

INTRODUCTION

Wavefield extrapolation extends surface recorded data to depth by application of a wave-equation operator. The choice of operator depends mainly on practical considerations (e.g. computer memory, total flop count); however, one persistent theoretical constraint is the degree of velocity model complexity. In laterally invariant media, closed-form Fourier-domain operators (single square root, SSR) can accurately extrapolate surface recorded wavefields up to 90° (Claerbout, 1985). However, such solutions are inapplicable in media characterized by lateral velocity variation, and approximate solutions to the SSR equation are employed. Consequently, the accuracy of the extrapolation operators degrades, particularly at high angles relative to the downward extrapolation axis, and more sophisticated procedures are required to ensure wavefield accuracy.

Improved wavefield extrapolation can be achieved in many ways. First, one may try to improve the high-angle accuracy of an operator while retaining a Cartesian computational grid. Examples of this approach include incorporating higher-order terms in the expansion of Fourier domain operators, e.g. Fourier finite-difference (Ristow and Ruhl, 1994), generalized screen propagator (de Hoop et al., 2000), or using tilted Cartesian coordinate systems (Zhang and McMechan, 1997; Etgen, 2002) that permit local extensions of accurate high-angle propagation. Second, seismic wavefields may be spatially partitioned into more manageable sections and then independently extrapolated in preferred directions. For example, one can decompose data sections to form local beams and extrapolate along tubes of finite thickness (Hill, 2001; Gray et al., 2002; Albertin et al., 2001; Brandsberg-Dahl and Etgen, 2003).

A third option is to abandon the strictures of Cartesian coordinates altogether and represent the physics of wavefield extrapolation in a generalized coordinate system that obeys the tenets of differential geometry (Guggenheimer, 1977). In particular, one could use a basis (or coordinate system) conforming to where wavefronts propagate. In this reference frame low-angle operators remain applicable, and the extrapolation procedure is of high fidelity, even at arbitrarily large angles to depth axis. The strategy espoused in this paper is the latter: it is more prudent to adjust the coordinate system to conform better with the physics than to force the physics to work in Cartesian coordinates, or on some aphysical spatial partition of the data.

One judicious choice of non-Cartesian coordinate system is a basis derived from a suite of rays. In this approach, the natural wavefield extrapolation direction is travel-time along a ray, with orthogonal coordinates directed across the rayfronts at a constant time step. However, unlike for the case of Cartesian coordinates, the distance between adjacent rays may freely expand or contract according to the lateral variations in the velocity model. Thus, properly defining the coordinate metric requires additional parameters that account for the Jacobian-like coordinate spreading. Given a rayfield and the associated Jacobian parameters, the solution to the corresponding one-way acoustic wave-equation is generated in an ordinary fashion. The ray-coordinate wavefield solution is then interpolated back to a Cartesian mesh through a simple mapping operation.

This paper presents a procedure for constructing a frequency dependent ray coordinate system in an adaptive manner directly from the wavefield. The key idea is that the rayfront vectors at any given step are directly calculable from the phase-gradient of previous wavefield solution steps. This naturally leads to a bootstrapping procedure where one alternates between calculating the coordinate system for the next step, and the corresponding wavefield solution at that step. The methodology is similar to the *Riemannian wavefield extrapolation* (RWE) technique presented by Sava and Fomel (2004), where rayfields are traced through a smoothed velocity model using a Huygens wavefront tracer (Sava and Fomel, 2001). The method differs, though, in that frequency stationarity of the rayfield is not assumed, and the rayfield is instead calculated directly from the monochromatic wavefield.

THEORY

Phase-rayfields

A monochromatic acoustic wavefield, $\mathcal{U}$, at frequency, ω, and spatial location, $\mathbf{x}$, may be represented by,

$$\mathcal{U}(\mathbf{x},\omega) = A(\mathbf{x},\omega)\,\mathrm{e}^{\mathrm{i}\phi(\mathbf{x},\omega)}, \tag{1}$$

where $A(\mathbf{x},\omega)$ and $\phi(\mathbf{x},w)$ are the amplitude and phase functions, respectively. For monochromatic waves propagating through isotropic media, the gradient of phase function, $\nabla\phi(\mathbf{x},w)$, represents the instantaneous direction of energy transport and is a characteristic to the solution of the governing Helmholtz equation (Foreman, 1989). Analogous to the ray precept in broadband theory, this vector quantity defines the instantaneous direction and magnitude of one ray in a continuous ray manifold. However, to differentiate between broadband and monochromatic ray representations, the latter are termed phase-rays. The governing differential equations for a phase-ray, r_i, are presented in the 'exact' ray formulation of Foreman (1989). In Cartesian coordinates, the subscript i on r refers to the projection of the ray along the x and z axes - r_x and r_z, respectively. The phase-ray equations, in summation notation, are,

$$\frac{\mathrm{d}r_i}{\mathrm{d}s} = \frac{\partial\phi}{\partial x_i}\left[\left(\frac{\partial\phi}{\partial x_k}\right)\left(\frac{\partial\phi}{\partial x_k}\right)\right]^{-\frac{1}{2}}, \tag{2}$$

where ϕ is the phase function, x_i is a coordinate of the underlying Cartesian grid, and the repeated index k represents a summation over all coordinate indices. Scalar step magnitude, $\mathrm{d}s$, is given by $\mathrm{d}s(\mathbf{x}) = v(\mathbf{x})\,\mathrm{d}\tau$, where $v(\mathbf{x})$ is the velocity in the neighborhood of ray, $r_i(\mathbf{x})$, and $\mathrm{d}\tau$ is an element of time along the ray.

Calculating phase-rays thus requires isolating the gradient of the monochromatic phase function. An efficient procedure is to calculate the ratio of the wavefield gradient to the wavefield itself

$$\frac{\nabla\mathcal{U}}{\mathcal{U}} = \frac{\nabla A}{A} + \mathrm{i}\nabla\phi, \tag{3}$$

and take the imaginary component,

$$\nabla\phi = \Im\left(\frac{\nabla\mathcal{U}}{\mathcal{U}}\right). \tag{4}$$

The right hand side of equation (4) is calculable only when a wavefield solution is known. The solution for a ray, r_i, is then computed by integrating the right hand side of equations (2). Ray solutions are uniquely determined given an initial starting position by reason that equations (2) form a decoupled system of differential equations of first-order. Accordingly, a phase-ray coordinate system is uniquely defined by specifying a set of initial points and a frequency, ω, and solving equations (2) with equations (3) and (4).

Ray-coordinate wavefield extrapolation

Wavefield extrapolation in ray-coordinates requires casting the acoustic wave-equation not in the usual Cartesian representation, but rather in a system parameterized by ray variables. In 2-D, these variables consist of τ, the one-way travel time from a source/receiver point along the direction of a ray, and γ, the direction across the rayfront at a constant time step.

The 2-D, ray-coordinate wave-equation for wavefield, $\mathcal{U}$, is,

$$\frac{1}{vJ}\left[\frac{\partial}{\partial\tau}\left(\frac{J}{v}\frac{\partial\mathcal{U}}{\partial\tau}\right)+\frac{\partial}{\partial\gamma}\left(\frac{v}{J}\frac{\partial\mathcal{U}}{\partial\gamma}\right)\right]=-\frac{\omega^2}{v^2}\mathcal{U}, \quad (5)$$

where v is the velocity function, and J is the ray-coordinate Jacobian or geometrical ray spreading factor given by $J=\sqrt{\frac{\partial x_k}{\partial\gamma}\frac{\partial x_k}{\partial\gamma}}$. Analogous to wave-equation extrapolation in Cartesian coordinates, a dispersion relation must be specified that forms the basis for all derived ray coordinate extrapolation operators. The relation being sought is the wavenumber along the ray direction, k_τ. Following Sava and Fomel (2004), the partial derivative operators in equation (5) are expanded out to generate a second-order partial differential equation with non-zero cross derivatives. Fourier domain wave-numbers are then substituted for the partial differential operators acting on wavefield, $\mathcal{U}$, and the quadratic formula is applied to yield the expression for k_τ,

$$\begin{aligned} k_\tau &= \frac{\mathrm{i}v}{2J}\frac{\partial}{\partial\tau}\left(\frac{J}{v}\right)\pm \\ &\pm \left[\omega^2-\left[\frac{v}{2J}\frac{\partial}{\partial\tau}\left(\frac{J}{v}\right)\right]^2+\frac{\mathrm{i}v}{J}\frac{\partial}{\partial\gamma}\left(\frac{v}{J}\right)k_\gamma-\frac{v^2k_\gamma^2}{J^2}\right]^{\frac{1}{2}} \end{aligned} \quad (6)$$

One relatively straightforward manner to apply wavenumber k_τ in an extrapolation scheme is to develop the ray-coordinate equivalent of Claerbout's classic 15° equation (Claerbout, 1985). This involves a second-order Taylor series expansion of the radical in equation (6), and the identification of Fourier dual parameters k_τ and k_γ with their space domain derivative counterparts $-\mathrm{i}\frac{\partial}{\partial\tau}$ and $-\mathrm{i}\frac{\partial}{\partial\gamma}$. The corresponding ray-coordinate 15° equation is,

$$\begin{aligned} \frac{\partial\mathcal{U}}{\partial\tau}\approx &-\frac{v}{2J}\frac{\partial}{\partial\tau}\left(\frac{J}{v}\right)+\mathrm{i}\omega_o+\frac{\mathrm{i}v}{2\omega_oJ}\frac{\partial}{\partial\gamma}\left(\frac{v}{J}\right)\frac{\partial\mathcal{U}}{\partial\gamma} \\ &+\frac{-\mathrm{i}}{2\omega_o}\left[\left(\frac{v}{2J\omega_o}\frac{\partial}{\partial\gamma}\left(\frac{v}{J}\right)\right)^2-\frac{v^2}{J^2}\right]\frac{\partial^2\mathcal{U}}{\partial\gamma^2}, \end{aligned} \quad (7)$$

where $\omega_o=\omega\sqrt{1-\left(\frac{v}{2\omega J}\frac{\partial}{\partial\tau}\left(\frac{J}{v}\right)\right)^2}$ may be considered as the effective frequency, Equation (7) may be solved in 2-D using fully implicit finite difference methods (e.g. Crank-Nicolson) and fast tridiagonal solvers.

Bootstrapping wavefield extrapolation

The 2-D phase-ray extrapolation approach detailed above is analogous to the fabled 'chicken and egg' conundrum: which to compute first? Stated explicitly, phase-rays must be calculated from a known wavefield solution; however, the wavefield is itself the quantity being computed. Because the wavefield is not known *a priori*, clearly a new strategy is required to resolve these disparate observations.

The approach described here is to use wavefields parameterized in phase-ray coordinates, rather than Cartesian, to dictate the direction of the next rayfront step. For judiciously chosen $\Delta\tau$ steps, both the rayfield direction and the resulting wavefield evolve slowly, and ray directions differ by only small, incremental, amounts in a neighborhood of τ. Hence, the rayfront vectors at any given step are directly calculable from the phase-gradient of previous wavefield solution steps. Thus, the phase-ray coordinate system may be computed on the fly using the wavefield phase gradient of the previous few steps and the local velocity.

Using a wavefield parameterized in ray coordinates to generate the underlying coordinate system requires that the governing phase-ray equations are transformed accordingly. Fortunately, the magnitude of a scalar field gradient remains invariant to coordinate transformation, and is related through, $\frac{\partial\phi}{\partial x_l}=\frac{\partial\phi}{\partial y_m}\frac{\partial y_m}{\partial x_l}$, where $x_l=[x,z]$ and $y_m=[\tau,\gamma]$ are the Cartesian and ray coordinate basis, respectively, and l and m are dummy indices. This reparameterization leads to the ray-coordinate phase-ray equations,

$$\frac{\mathrm{d}r_i}{\mathrm{d}s}=\frac{\partial\phi}{\partial y_j}\frac{\partial y_j}{\partial x_i}\left[\left(\frac{\partial\phi}{\partial y_m}\frac{\partial y_m}{\partial x_l}\right)\left(\frac{\partial\phi}{\partial y_m}\frac{\partial y_m}{\partial x_l}\right)\right]^{-\frac{1}{2}}. \quad (8)$$

The partial derivatives between the two coordinate systems are directly related to traditional ray parameters. Cartesian derivatives of τ are the horizontal and vertical plane wave slownesses, while those with respect to γ are the local rotation angle to the Cartesian coordinate system. Explicitly, these functions are,

$$\frac{\partial\tau}{\partial x}=\frac{\sin\theta}{v(\mathbf{x})},\quad \frac{\partial\tau}{\partial z}=\frac{\cos\theta}{v(\mathbf{x})},\quad \frac{\partial\gamma}{\partial x}=\cos\theta,\quad \frac{\partial\gamma}{\partial z}=\sin\theta \quad (9)$$

where parameter θ is the angle formed between a ray and the z-axis.

Having parameterized the phase-ray equations in ray-coordinates, and specified how to update rayfront directions, it is possible to detail the bootstrap method that is central to the adaptive phase-ray extrapolation procedure. The coordinate system is first initialized by assuming the first $M+1$ steps using an educated guess of where the wavefront will propagate. M wavefield extrapolation steps are then carried out to generate the starting wavefield. The bootstrap process then consists of a loop around three separate calculations: i) ray step Δr_i from the previous M wavefield steps; ii) rayfield Jacobian spreading and associated functions; and iii) wavefield $\mathcal{U}$. The final step involves interpolating the wavefield from ray to Cartesian coordinates.

PHASE-RAY EXTRAPOLATION EXAMPLES

The utility of the adaptive phase-ray wavefield extrapolation method is illustrated with 2-D synthetic examples involving progressively more complex velocity models. Underlying ray coordinate systems were calculated according to equation (8), and wavefield were extrapolated on the computed rayfields using the ray coordinate 15° equation (equation 7). All images were computed using uniform time steps with individual $\Delta\tau$ dependent on velocity model complexity. Orthogonal coordinate γ was parameterized as either a punctual or a plane wave source. Point source images required a parameterization of γ over shooting angle starting at radius, R. Initial wavefields consisted of constant amplitude lines separated in τ and distributed uniformly over coordinate γ. Plane-wave images required a parameterization of γ over surface coordinate x_o. Initial wavefields again consisted of constant amplitude lines separated in τ and distributed uniformly over surface position, x_o.

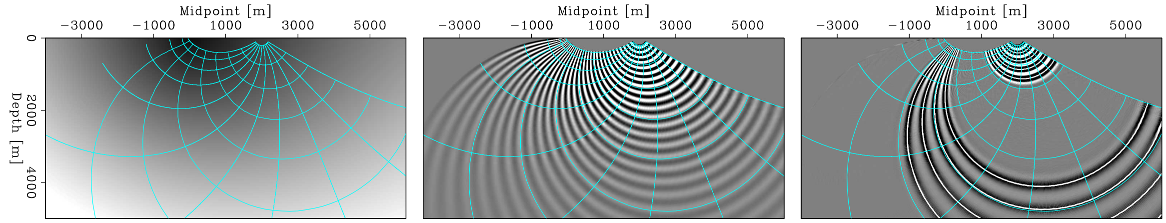

Figure 1: Point source extrapolation example. Left: velocity model with 5Hz ray coordinate system. Middle: 5Hz monochromatic wavefield. Right: broadband (0.2-25Hz) wavefield.

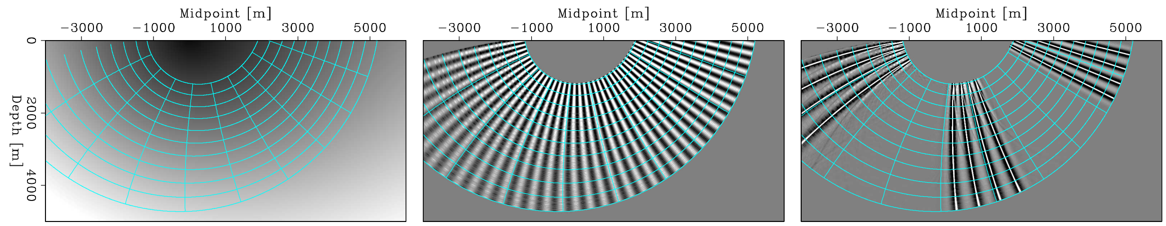

Figure 2: Plane wave extrapolation example. Left: velocity model with 5Hz ray coordinate system. Middle: 5Hz monochromatic wavefield. Right: Broadband (0.2-25Hz) wavefield.

The first extrapolation example was designed to test the method on a smooth velocity function of sufficient contrast to enable overturning waves. The velocity model, shown in the left panel of figure (1), consists of a broad Gaussian velocity anomaly that is 86% slower than the background velocity of 3000m/s. Superposed on the Gaussian anomaly are vertical and horizontal gradients of 0.1 and -0.05km/s per km, respectively. Phase-ray wavefield extrapolation was carried out at 5Hz. The resulting ray coordinate system (left panel) is smooth and triplication-free. The corresponding 5Hz wavefield (middle panel) was interpolated from ray coordinates to a Cartesian mesh. The broadband wavefield (0.2-25Hz) (right panel) was computed holding the ray coordinate system stationary at 5Hz. These these panels demonstrate the successful overturning of both the rayfield and wavefield.

The second example, shown in figure (2), is the plane-wave equivalent to figure (1). The initial coordinate system and wavefield tilt angle was -10°. The left and middle panels show the 5Hz rayfield and the corresponding 5Hz wavefield, respectively. The right panel presents the broadband wavefield calculated in the 0.2-25Hz frequency band on a ray coordinate system held stationary at 5Hz.

The next example consists of a Gulf of Mexico salt model (figure 3). The background velocity of the model is a typical Gulf of Mexico v(z) velocity gradient. The superposed salt body is characterized by higher wave speeds (4700m/s) and a somewhat rugose bottom of salt interface. The initial angular coverage was restricted to where the ray-coordinate system did not triplicate. The middle and right panels present the 5Hz monochromatic and 2-35Hz broadband wavefields, respectively. Again, the ray coordinate system in the right panel was held stationary at 5Hz. The two wavefields (middle and right panels) demonstrate the effect of strong velocity contrasts and a rugose interface between the salt body and the enveloping sediments. At angles tending away from vertical, the wavefield increasingly refracts in accordance with Snell's law, becomes horizontal, impinges on the salt-sediment interface, and eventually refracts upward at fairly steep angles.

RAY-COORDINATE TRIPLICATION

The phase-ray extrapolation examples discussed so far have intentionally avoided triplicating wavefields. However, wavefield triplications are commonly observed in seismic data, especially in areas of complex geology where the phase-ray extrapolation technique shows most potential. In this formulation, the ray coordinate system is computed from previous wavefield steps; hence, the underlying basis will begin to triplicate immediately after the wavefield does. Thus, a contingency plan must exist to prevent numerical instabilities associated with coordinate triplication.

Wavefield triplication occurs when a propagating wavefield is focused by lateral velocity variation acting as an optic lens. One canonical example is a Gaussian-shaped slow velocity anomaly, where continued wavefields exhibit a characteristic bow-tie signature beneath the anomaly. Numerical instabilities occur when calculating the ray coordinate system in the vicinity of the bow-tie because neighboring rays overlap while following their respective bow-tie branches. At the crossing point, the Jacobian is identically zero leading to infinite values of wavenumber k_τ. Infinite wavenumbers are not realizable in practice and are only a theoretical artifice of the wavefield being multivalued at that point. Accordingly, instabilities with ray coordinate triplication may be rectified through an appropriate accounting for wavefield multivaluedness.

One way to deal with multivalued functions is to treat the individual branches of the triplication bow-tie as independent wavefield components that should be held incommunicado. This requires computing the location(s) of wavefield triplications (i.e. crossing ray segments) from the rayfield. In 2-D, crossing ray segments may be identified by computing their intersection point, and testing if this location falls in the area bounded by the ray segments. Where this test reveals a crossing point the rayfield has triplicated and should be cut into individual branches. Functions requiring the computation of derivatives are then calculated on their respective branches. For locations not on branch cuts, centered finite-difference stencils may be used; however, at branch-cut locations appropriate left- and right-sided stencils are required. The locations of branch cuts are kept for all subsequent computations. A more general formulation involving the oriented wave equation (Fomel, 2003) has the potential to address this problem in a robust theoretical framework.

The canonical example of a slow Gaussian-shaped velocity anomaly is presented in figure (4). The velocity model (left panel) consists of a slow Gaussian anomaly of maximum -50% perturbation of the 2000m/s background velocity. The 10Hz phase-ray coordinate sys-

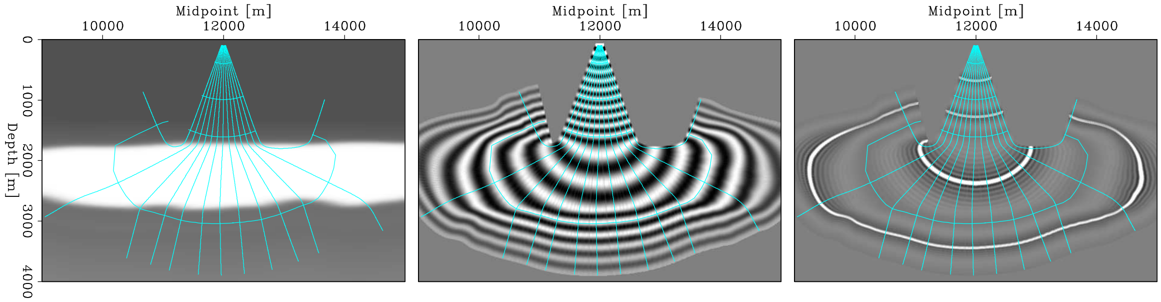

Figure 3: Point source, Gulf of Mexico salt example. Left: velocity model and 5Hz monochromatic rayfield. Middle: 5Hz wavefield. Right: broadband (2-35Hz) wavefield.

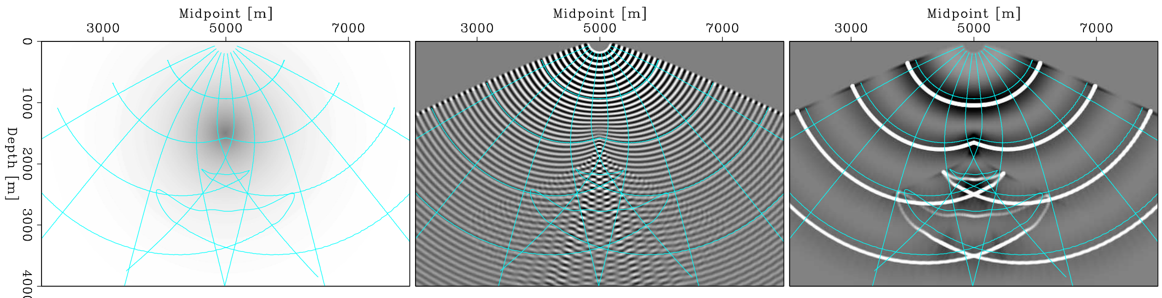

Figure 4: Slow Gaussian slow anomaly example. Left: model with 10Hz coordinate system. Middle: 10Hz wavefield. Right: broadband (0.1-30Hz) wavefield.

tem is also overlain. The 10Hz wavefield (middle panel) shows a hatched pattern created by the superposition of the phases of the two competing triplication branches. The right panel presents the broadband result (0.1-30Hz) computed on a stationary 10Hz coordinate system.

CONCLUSIONS

The utility of a ray-based RWE scheme is demonstrated using an adaptive ray coordinate system computed directly from the wavefield. We present a bootstrap procedure that allows for an adaptive, frequency-dependent coordinate system to be computed on the fly from the wavefield phase gradient and the velocity model. Coupling this procedure with RWE leads to a general frequency-dependent extrapolation procedure capable of following a wavefront as it propagates, overturns, and even triplicates.

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